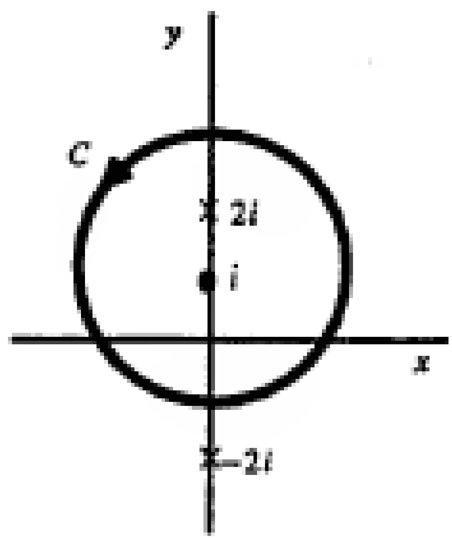
MATH 2230 Solution for Tutorial Question 10

1. Let C denote the positively oriented circle $|z - i| = 2$, shown below.

Graph



Evaluate following integrals

(a)

$$\int_C \frac{1}{z^2+4}dz$$

(b)

$$\int_C \frac{1}{(z^2+4)^2}dz$$

Solution

The Cauchy integral formula enables us to write

$$\int_C \frac{1}{z^2+4}dz = \int_C \frac{1}{(z+2i)(z-2i)}dz = 2\pi i\Big[\frac{1}{z+2i}\Big]_{z=2i} = \frac{\pi}{2}.$$

Applying the extended form of the Cauchy integral formula, we have

$$\begin{aligned}\int_C \frac{1}{(z^2+4)^2}dz &= \int_C \frac{1}{(z+2i)^2(z-2i)^2}dz \\ &= \frac{2\pi i}{1!}\Big[\frac{d}{dz}\frac{1}{(z+2i)^2}\Big]_{z=2i} \\ &= 2\pi i\Big[\frac{-2}{(z+2i)^3}\Big]_{z=2i} = \frac{\pi}{16}\end{aligned}$$

2. Derive the expansions

(a)
$$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty);$$

(b)
$$\frac{\sin z^2}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \quad (0 < |z| < \infty).$$

Solution Recall the expansion

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \quad (|z| < \infty)$$

Then we have

$$\begin{aligned} \frac{\sinh z}{z^2} &= \frac{1}{z} + \frac{z}{3!} + \frac{z^3}{5!} + \cdots \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty). \end{aligned}$$

Since

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \quad (|z| < \infty)$$

Replacing z by z^2 yields

$$\sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots \quad (|z| < \infty)$$

Hence

$$\frac{\sin z^2}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \quad (0 < |z| < \infty).$$

3. Give two Laurent series expansions in powers of z for the functions

(a)
$$f(z) = \frac{1}{z^2(1-z)}$$

(b)
$$f(z) = \frac{1}{z(1+z^2)}$$

and specify the regions in which those expansion are valid.

Solution

(a) The function $\dfrac{1}{z^2(1-z)}$ has singularities $z=0$ and $z=1$. Hence there are Laurent series in powers of z for the domains $0<|z|<1$ and $1<|z|<\infty$. To find the series when $0<|z|<1$, recall that

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n, \quad |z|<1$$

and write

$$f(z)=\frac{1}{z^2}\sum_{n=0}^{\infty} z^n=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty} z^n, \quad 0<|z|<1.$$

When $1<|z|<\infty$, we have $\frac{1}{|z|}<1$ and

$$f(z)=-\frac{1}{z^3}\frac{1}{1-\frac{1}{z}}=-\frac{1}{z^3}\sum_{n=0}^{\infty}(\frac{1}{z})^n=-\sum_{n=3}^{\infty}\frac{1}{z^n}, \quad 1<|z|<\infty.$$

(b) The function $f(z)=\dfrac{1}{z(1+z^2)}$ has singularities $z=0$ and $z=\pm i$. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$. For the domain $0<|z|<1$

$$f(z)=\frac{1}{z(1+z^2)}=\frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}.$$

On the other hand, when $1<|z|<\infty$,

$$f(z)=\frac{1}{z(1+z^2)}=\frac{1}{z^3}\sum_{n=0}^{\infty}(-\frac{1}{z^2})^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}.$$